Sixth Term Examination Papers 9465

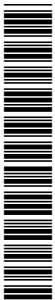## MATHEMATICS 1
Morning

FRIDAY 24 JUNE 2011
Time: 3 hours

Additional Materials: Answer Booklet
Formulae Booklet

INSTRUCTIONS TO CANDIDATES

Please read this page carefully, but do not open this question paper until you are told that you may do so.

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

Write the numbers of the questions you answer in the order attempted on the front of the answer booklet.

INFORMATION FOR CANDIDATES

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

Calculators are not permitted.

Please wait to be told you may begin before turning this page.

This question paper consists of 7 printed pages and 1 blank page.

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Section A: Pure Mathematics

1   **(i)**   Show that the gradient of the curve $\dfrac{a}{x} + \dfrac{b}{y} = 1$, where $b \neq 0$, is $-\dfrac{ay^2}{bx^2}$.

The point (p, q) lies on both the straight line $ax + by = 1$ and the curve $\dfrac{a}{x} + \dfrac{b}{y} = 1$, where $ab \neq 0$. Given that, at this point, the line and the curve have the same gradient, show that $p = \pm q$.

Show further that either $(a - b)^2 = 1$ or $(a + b)^2 = 1$.

(ii) Show that if the straight line $ax + by = 1$, where $ab \neq 0$, is a normal to the curve $\dfrac{a}{x} - \dfrac{b}{y} = 1$, then $a^2 - b^2 = \frac{1}{2}$.

2   The number E is defined by $E = \displaystyle\int_0^1 \frac{e^x}{1+x}\, dx$.

Show that

$$\int_0^1 \frac{xe^x}{1+x}\, dx = e - 1 - E,$$

and evaluate $\displaystyle\int_0^1 \frac{x^2 e^x}{1+x}\, dx$ in terms of e and E.

Evaluate also, in terms of E and e as appropriate:

(i) $\displaystyle\int_0^1 \frac{e^{\frac{1-x}{1+x}}}{1+x}\, dx$;

(ii) $\displaystyle\int_1^{\sqrt{2}} \frac{e^{x^2}}{x}\, dx$.

3 Prove the identity

$$4\sin\theta\sin(\tfrac{1}{3}\pi - \theta)\sin(\tfrac{1}{3}\pi + \theta) = \sin 3\theta.$$ (∗)

(i) By differentiating (∗), or otherwise, show that

$$\cot\tfrac{1}{9}\pi - \cot\tfrac{2}{9}\pi + \cot\tfrac{4}{9}\pi = \sqrt{3}.$$

(ii) By setting $\theta = \tfrac{1}{6}\pi - \phi$ in (∗), or otherwise, obtain a similar identity for $\cos 3\theta$ and deduce that

$$\cot\theta\cot(\tfrac{1}{3}\pi - \theta)\cot(\tfrac{1}{3}\pi + \theta) = \cot 3\theta.$$

Show that

$$\operatorname{cosec}\tfrac{1}{9}\pi - \operatorname{cosec}\tfrac{5}{9}\pi + \operatorname{cosec}\tfrac{7}{9}\pi = 2\sqrt{3}.$$

4 The distinct points P and Q, with coordinates $(ap^2, 2ap)$ and $(aq^2, 2aq)$ respectively, lie on the curve $y^2 = 4ax$. The tangents to the curve at P and Q meet at the point T. Show that T has coordinates $\big(apq, a(p+q)\big)$. You may assume that $p \neq 0$ and $q \neq 0$.

The point F has coordinates $(a, 0)$ and ϕ is the angle TFP. Show that

$$\cos\phi = \frac{pq + 1}{\sqrt{(p^2 + 1)(q^2 + 1)}}$$

and deduce that the line FT bisects the angle PFQ.

5 Given that $0 < k < 1$, show with the help of a sketch that the equation

$$\sin x = kx$$ (∗)

has a unique solution in the range $0 < x < \pi$.

Let

$$I = \int_0^\pi \big|\sin x - kx\big|\,dx.$$

Show that

$$I = \frac{\pi^2\sin\alpha}{2\alpha} - 2\cos\alpha - \alpha\sin\alpha,$$

where α is the unique solution of (∗).

Show that I, regarded as a function of α, has a unique stationary value and that this stationary value is a minimum. Deduce that the smallest value of I is

$$-2\cos\frac{\pi}{\sqrt{2}}.$$

6 Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.

(i) Show that the coefficient of x^r in the expansion of

$$\frac{1-x+2x^2}{(1-x)^3}$$

is r^2+1 and hence find the sum of the series

$$1 + \frac{2}{2} + \frac{5}{4} + \frac{10}{8} + \frac{17}{16} + \frac{26}{32} + \frac{37}{64} + \frac{50}{128} + \cdots .$$

(ii) Find the sum of the series

$$1 + 2 + \frac{9}{4} + 2 + \frac{25}{16} + \frac{9}{8} + \frac{49}{64} + \cdots .$$

7 In this question, you may assume that $\ln(1+x) \approx x - \frac{1}{2}x^2$ when $|x|$ is small.

The height of the water in a tank at time t is h. The initial height of the water is H and water flows into the tank at a constant rate. The cross-sectional area of the tank is constant.

(i) Suppose that water leaks out at a rate proportional to the height of the water in the tank, and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(\alpha^2 H - h),$$

for some positive constant k. Deduce that the time T taken for the water to reach height αH is given by

$$kT = \ln\left(1 + \frac{1}{\alpha}\right),$$

and that $kT \approx \alpha^{-1}$ for large values of α.

(ii) Suppose that the rate at which water leaks out of the tank is proportional to $\sqrt{h}$ (instead of h), and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that the time T' taken for the water to reach height αH is given by

$$cT' = 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha\ln\left(1 + \frac{1}{\sqrt{\alpha}}\right)\right)$$

for some positive constant c, and that $cT' \approx \sqrt{H}$ for large values of α.

8 **(i)** The numbers m and n satisfy

$$m^3 = n^3 + n^2 + 1.$$ $(*)$

    (a) Show that $m > n$. Show also that $m < n+1$ if and only if $2n^2 + 3n > 0$. Deduce that $n < m < n+1$ unless $-\frac{3}{2} \leqslant n \leqslant 0$.

    (b) Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m, n) = (1, 0)$ and $(m, n) = (1, -1)$.

  (ii) Find all integer solutions of the equation

$$p^3 = q^3 + 2q^2 - 1.$$

Section B: Mechanics

9 A particle is projected at an angle θ above the horizontal from a point on a horizontal plane. The particle just passes over two walls that are at horizontal distances d_1 and d_2 from the point of projection and are of heights d_2 and d_1, respectively. Show that

$$\tan \theta = \frac{d_1^2 + d_1 d_2 + d_2^2}{d_1 d_2} \, .$$

Find (and simplify) an expression in terms of d_1 and d_2 only for the range of the particle.

10 A particle, A, is dropped from a point P which is at a height h above a horizontal plane. A second particle, B, is dropped from P and first collides with A after A has bounced on the plane and before A reaches P again. The bounce and the collision are both perfectly elastic. Explain why the speeds of A and B immediately before the first collision are the same.

The masses of A and B are M and m, respectively, where $M > 3m$, and the speed of the particles immediately before the first collision is u. Show that both particles move upwards after their first collision and that the maximum height of B above the plane after the first collision and before the second collision is

$$h + \frac{4M(M - m)u^2}{(M + m)^2 g} \, .$$

11 A thin non-uniform bar AB of length $7d$ has centre of mass at a point G, where $AG = 3d$. A light inextensible string has one end attached to A and the other end attached to B. The string is hung over a smooth peg P and the bar hangs freely in equilibrium with B lower than A. Show that
$$3 \sin \alpha = 4 \sin \beta \, ,$$
where α and β are the angles PAB and PBA, respectively.

Given that $\cos \beta = \frac{4}{5}$ and that α is acute, find in terms of d the length of the string and show that the angle of inclination of the bar to the horizontal is $\arctan \frac{1}{7}$.

Section C: Probability and Statistics

12 I am selling raffle tickets for £1 per ticket. In the queue for tickets, there are m people each with a single £1 coin and n people each with a single £2 coin. Each person in the queue wants to buy a single raffle ticket and each arrangement of people in the queue is equally likely to occur. Initially, I have no coins and a large supply of tickets. I stop selling tickets if I cannot give the required change.

(i) In the case $n = 1$ and $m \geqslant 1$, find the probability that I am able to sell one ticket to each person in the queue.

(ii) By considering the first three people in the queue, show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 2$ and $m \geqslant 2$ is $\dfrac{m-1}{m+1}$.

(iii) Show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 3$ and $m \geqslant 3$ is $\dfrac{m-2}{m+1}$.

13 In this question, you may use without proof the following result:

$$\int \sqrt{4 - x^2}\,\mathrm{d}x = 2\arcsin(\tfrac{1}{2}x) + \tfrac{1}{2}x\sqrt{4 - x^2} + c\,.$$

A random variable X has probability density function f given by

$$\mathrm{f}(x) = \begin{cases} 2k & -a \leqslant x < 0 \\[2mm] k\sqrt{4 - x^2} & 0 \leqslant x \leqslant 2 \\[2mm] 0 & \text{otherwise,} \end{cases}$$

where k and a are positive constants.

(i) Find, in terms of a, the mean of X.

(ii) Let d be the value of X such that $\mathrm{P}(X > d) = \tfrac{1}{10}$. Show that $d < 0$ if $2a > 9\pi$ and find an expression for d in terms of a in this case.

(iii) Given that $d = \sqrt{2}$, find a.

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